数理解析研究所講究録
第 1569 巻 2007 年 52-58

Construction of smooth actions on spheres for Smith equivalent representations

岡山大学大学院自然科学研究科　森本 雅治 (MORIMOTO, MASAHARU)
Graduate School of Natural Science and Technology
Okayama University

1. PROBLEMS AND RESULTS

Throughout this paper, let G be a finite group. A real G-representation of finite dimension is meant by a *real G-module*, a smooth manifold is meant by a *manifold*, and a smooth G-manifold is meant by a *G-manifold*. For a G-manifold X, let $\mathcal{TR}(X)$ denote the set of all isomorphism classes (as real G-modules) of tangential representations $T_x(X)$, where x runs over the G-fixed point set X^G. We are interested in $\mathcal{TR}(X)$ for manifolds X such that X^G consists of exactly two points. In particular, the case where X are homotopy spheres has been studied as Smith Problem.

Smith Problem. Let Σ be a homotopy sphere with G-action such that the G-fixed point set consists of exactly two points a, b. Are the tangential representations $T_a(\Sigma)$ and $T_b(\Sigma)$ isomorphic to each other (namely $|\mathcal{TR}(\Sigma)| = 1$) ?

We have affirmative answers (e.g. Atiyah-Bott, Milnor, Sanchez) as well as negative answers (e.g. Petrie, Cappell-Shaneson, Petrie-Randall, Petrie-Dovermann, Dovermann-Washington, Dovermann-Suh, Laitinen-Pawałowski, Pawałowski-Solomon), to Smith Problem under various hypotheses. There are surveys relevant to studies on Smith Problem in [24], [18] and [6].

To study the problem, we define the following relations $\sim_{\mathfrak{D}}$, $\sim_{\mathfrak{S}}$ and $\sim_{\mathfrak{D}\mathfrak{S}}$. In the definition below, V and W are real G-modules.

(1) $V \sim_{\mathfrak{D}} W$ if there exists a disk D with G-action such that $D^G = \{a, b\}$ and $\{[V], [W]\} = \mathcal{TR}(D)$.
(2) $V \sim_{\mathfrak{S}} W$ if there exists a homotopy sphere Σ with G-action such that $\Sigma^G = \{a, b\}$ and $\{[V], [W]\} = \mathcal{TR}(\Sigma)$.
(3) $V \sim_{\mathfrak{D}\mathfrak{S}} W$ if $V \sim_{\mathfrak{D}} W$ and $V \sim_{\mathfrak{S}} W$.

2000 Mathematics Subject Classification. Primary: 57S17, 57S25, 55M35. Secondary: 20C05.

Here $\sim_{\mathfrak{D}}$ and $\sim_{\mathfrak{D}\mathfrak{S}}$ may not be equivalence relations, although they stably yield equivalence relations. We have been interested in the relation $\sim_{\mathfrak{S}}$ (namely the Smith equivalence), but in the present paper we will mainly pay our attention to the relation $\sim_{\mathfrak{D}\mathfrak{S}}$.

Let $\mathrm{RO}(G)$ denote the real representation ring. We define the subsets $\mathfrak{D}(G)$, $\mathfrak{S}(G)$ and $\mathfrak{D}\mathfrak{S}(G)$ of $\mathrm{RO}(G)$ by

$$\mathfrak{D}(G) = \{V - W \in \mathrm{RO}(G) \mid V \sim_{\mathfrak{D}} W\}$$
$$\mathfrak{S}(G) = \{V - W \in \mathrm{RO}(G) \mid V \sim_{\mathfrak{S}} W\}$$
$$\mathfrak{D}\mathfrak{S}(G) = \mathfrak{D}(G) \cap \mathfrak{S}(G)$$

The set $\mathfrak{S}(G)$ was usually denoted by $Sm(G)$. By R. Oliver [16], there exists a disk with G-action with $|D^G| = 2$ if and only if G is an Oliver group (namely, G is not a mod $\mathcal{P}$ hyperelementary group). Thus it is worthwhile to study $\mathfrak{D}(G)$ and $\mathfrak{D}\mathfrak{S}(G)$ only for Oliver groups G.

If M is a subset of $\mathrm{RO}(G)$ then for families $\mathcal{A}$, $\mathcal{B}$ consisting of subgroups of G we define

$$M_{\mathcal{A}} \stackrel{def}{=} \{x \in M \mid \mathrm{res}^G_H x = 0 \ \forall\ H \in \mathcal{A}\}$$
$$M^{\mathcal{B}} \stackrel{def}{=} \{x = V - W \in M \mid V^K = 0 = W^K \ \forall\ K \in \mathcal{B}\}$$
$$M^{\mathcal{B}}_{\mathcal{A}} \stackrel{def}{=} \{x = V - W \in M_{\mathcal{A}} \mid V^K = 0 = W^K \ \forall\ K \in \mathcal{B}\}.$$

Using the notation with the families

$$\mathcal{P} = \mathcal{P}(G) \stackrel{def}{=} \{P \leq G \mid |P| = p^a \ (p \text{ a prime})\}$$
$$\mathcal{N}_2 = \mathcal{N}_2(G) \stackrel{def}{=} \{N \trianglelefteq G \mid |G/N| = 1,\ 2\}$$
$$\mathcal{N} = \mathcal{N}(G) \stackrel{def}{=} \{N \trianglelefteq G \mid |G/N| = 1 \text{ or a prime}\}$$
$$\mathcal{L} = \mathcal{L}(G) \stackrel{def}{=} \{L \leq G \mid L \supseteq G^{\{p\}} \text{ for some prime } p\},$$

we study the subsets $\mathfrak{D}(G)$, $\mathfrak{S}(G)$ and $\mathfrak{D}\mathfrak{S}(G)$ of $\mathrm{RO}(G)$. Here the group $G^{\{p\}}$ is the smallest normal subgroup of G with prime power index, namely

$$G^{\{p\}} = \bigcap_{H \trianglelefteq G : |G/H| = p^a \text{ for some } a} H.$$

An element in $\mathcal{L}$ defined above is called a *large subgroup* of G.

Many authors (e.g. Petrie-Randall, Petrie-Dovermann, Dovermann-Washington, Dovermann-Suh, Laitinen-Pawałowski, Pawałowski-Solomon) found various pairs (V, W) of nonisomorphic $\mathfrak{D}\mathfrak{S}$-related real G-modules V, W. But their (V, W) with $V \sim_{\mathfrak{D}\mathfrak{S}} W$ satisfy $V^N = 0 = W^N$ for all $N \triangleleft G$ with prime index. In other words, they showed

$$\mathfrak{D}\mathfrak{S}(G)^{\mathcal{N}} \neq 0$$

for various G. Now we recall the next proposition.

Proposition 1 ([12], [13]). *The implications* $\mathfrak{S}(G) \subseteq \mathrm{RO}(G)^{\mathcal{N}_2}_{\mathcal{Q}}$ *and* $\mathfrak{D}\mathfrak{S}(G) \subseteq \mathrm{RO}(G)^{\mathcal{N}_2}_{\mathcal{P}}$ *hold.*

These facts motivate us to study the following problem.

Problem A. Does there exist a finite group G satisfying $\mathfrak{D}\mathfrak{S}(G) \neq \mathfrak{D}\mathfrak{S}(G)^{\mathcal{N}}$?

The notion *gap module* is convenient to study this problem as well as Smith Problem. A real G-module V is called a *gap module* if it satisfies the following conditions.

(1) $V^L = 0$ for all $L \in \mathcal{L}(G)$.

(2) $\dim V^P > \dim V^H$ for all pairs (P, H) of subgroups of G such that $P \in \mathcal{P}(G)$ and $H > P$.

A finite group G is called a *gap group* if G admits a gap real G-module. Pawałowski-Solomon showed in [18] that for an arbitrary nonsolvable gap group G with $a_G \geq 2$ and $G \not\cong P\Sigma L(2, 27)$,

$$\mathfrak{D}\mathfrak{S}(G) \supseteq \mathrm{RO}(G)^{\mathcal{L}}_{\mathcal{P}} \neq 0.$$

Since the appearance of this result, the next problem has been asked.

Problem B. Are the sets $\mathfrak{S}(G)$ and $\mathfrak{D}\mathfrak{S}(G)$ nontrivial in the case $G = P\Sigma L(2, 27)$?

The purpose of the present paper is to answer to Problems A and B, and we obtained the following results.

Theorem 2. *For each odd prime p, there exist a gap Oliver group G and real G-modules V and W such that $V \sim_{\mathfrak{D}\mathfrak{S}} W$, $\dim V^N > 0$ and $\dim W^N = 0$ for some $N \lhd G$ with $|G/N| = p$, hence $\mathfrak{D}\mathfrak{S}(G) \neq \mathfrak{D}\mathfrak{S}(G)^{\mathcal{N}}$.*

Let $SG(m, n)$ denote the small group of order m and type n appearing in the computer software GAP [5].

Theorem 3. *Let $G = P\Sigma L(2, 27)$, $SG(864, 2666)$, or $SG(864, 4666)$. Then* $\mathrm{RO}(G)^{\mathcal{L}}_{\mathcal{P}} = 0$ *but*

$$\mathfrak{S}(G) = \mathfrak{D}(G) = \mathfrak{D}\mathfrak{S}(G) = \mathrm{RO}(G)^{\{G\}}_{\mathcal{P}} \cong \mathbb{Z}.$$

2. Additional Information

For $g \in G$, let (g) denote the conjugacy class of g in G. The *real conjugacy class* $(g)^{\pm}$ of g is the union of (g) and (g^{-1}). Let a_G denote the number of all real conjugacy classes of elements g of G such that g does not have prime power order. By the representation theory, we have

$$a_G = \text{rank } \mathrm{RO}(G)_{\mathcal{P}}.$$

Let δ denote the homomorphism from $\mathrm{RO}(G)_{\mathcal{P}}$ to $\mathbb{Z}$ given by

$$\delta([V]-[W]) = \dim V^G - \dim W^G.$$

Then by definition,

$$\mathrm{RO}(G)_{\mathcal{P}}^{\{G\}} = \mathrm{Ker}[\delta : \mathrm{RO}(G)_{\mathcal{P}} \to \mathbb{Z}].$$

B. Oliver [17] showed that if $a_G \geq 1$ then

$$\mathrm{Image}[\delta : \mathrm{RO}(G)_{\mathcal{P}} \to \mathbb{Z}] \supseteq 2\mathbb{Z}.$$

Thus the next proposition immediately follows.

Proposition (Laitinen-Pawałowski [8]). *If* $a_G \geq 1$ *then* rank $\mathrm{RO}(G)_{\mathcal{P}}^{\{G\}} = a_G - 1$.

In addition, B. Oliver [17] implies the next result.

Theorem (Oliver). *If G is an Oliver group then* $\mathfrak{D}(G) = \mathrm{RO}(G)_{\mathcal{P}}^{\{G\}}$.

Viewing these facts, E. Laitinen conjectured the next.

Laitinen's Conjecture. If G is an Oliver group with $a_G \geq 2$ then $\mathfrak{DS}(G) \neq 0$.

This conjecture had been positively expected until 2006. We, however, have a negative example.

Theorem 4 ([12], [13]). *Let* $G = \mathrm{Aut}(A_6)$. *Then Laitinen's Conjecture fails, in fact* $a_G = 2$ *and* $\mathfrak{S}(G) = 0 = \mathfrak{DS}(G)$.

Most finite Oliver groups are gap groups, but neither S_5 nor $\mathrm{Aut}(A_6)$ is a gap group, where S_5 is the symmetric group on five letters and A_6 is the alternating group on six letters. Pawałowski-Solomon [18] showed the next theorem using a deleting-inserting theorem of G-fixed point sets to disks ([10], [15, Appendix]).

Theorem (Pawałowski-Solomon [18]). *If G is a gap Oliver group then*

$$\mathrm{RO}(G)_{\mathcal{P}}^{\mathcal{L}} \subseteq \mathfrak{DS}(G).$$

On the other hand, they also showed the next result using the finite group theory.

Theorem (Pawałowski-Solomon [18]). *Let G be a nonsolvable gap group with* $a_G \geq 2$. *If* $G \not\cong P\Sigma L(2,27)$ *then*

$$\mathrm{RO}(G)_{\mathcal{P}}^{\mathcal{L}} \neq 0.$$

Putting these results together, we obtain a corollary.

Corollary (Pawałowski-Solomon [18]). *Let G be a nonsolvable gap group with $a_G \geq 2$. If $G \not\cong P\Sigma L(2,27)$ then $\mathfrak{DS}(G) \neq 0$.*

Since $S_5 \times C_2$, where C_2 is the cyclic group of order 2, is not a gap group, the next result is also interesting.

Theorem (X.M. Ju [6]). *In the case $G = S_5 \times C_2$, the equalities*

$$\mathfrak{S}(G) = \mathfrak{DS}(G) = \mathrm{RO}(G)^{\mathcal{L}}_{\mathcal{P}} \cong \mathbb{Z}$$

hold.

We obtained a deleting-inserting theorem [14] of new kind by employing an equivariant interpretation of Cappell-Shaneson's surgery obstruction theory for getting homology (possibly, not homotopy) equivalences as well as employing the induction theory of Wall's surgery obstruction groups. We state here the theorem in a simplified form.

Theorem 5. *Let G be an Oliver group and Y a disk with G-action. Suppose the following conditions are satisfied.*

(1) $Y^G = \{y_1, \ldots, y_m\}$, *where* $m \geq 1$.
(2) $\partial Y^L = \emptyset$ *for all* $L \in \mathcal{L}(G)$.
(3) $\dim Y^H \geq 5$ *for all mod $\mathcal{P}$ cyclic subgroups H, i.e.* $1 \underset{\mathcal{P}}{\triangleleft} P \underset{\text{cyclic}}{\triangleleft} H$.
(4) $\dim Y^P > 2(\dim Y^H + 1)$ *for all* $P \in \mathcal{P}(G)$ *and* $H > P$.
(5) $|\pi_1(Y^P)| < \infty$ *and* $(|\pi_1(Y^P)|, |P|) = 1$ *for all* $P \in \mathcal{P}(G)$.
(6) *The inclusion induced maps* $\pi_1(\partial Y^P) \to \pi_1(Y^P)$ *are isomorphisms for all* $P \in \mathcal{P}(G)$.

Then there exists a disk X with G-action such that $\partial X = \partial Y$ and $X^G = \emptyset$.

Remark that the union $\Sigma = X \cup_\partial Y$ identified along the boundaries of X and Y in the theorem above is a homotopy sphere such that $\mathcal{TR}(\Sigma) = \mathcal{TR}(Y)$. Since various G-actions on disks Y are constructed by Oliver's theory [17], we would obtain G-actions on homotopy spheres Σ from those on disks. In fact, the next result is an outcome of Theorem 5.

Theorem 6. *Let p be an odd prime. Let G be an Oliver group such that $G = G^{\{q\}}$ for all primes $q \neq p$ and $|G/G^{\{p\}}| = p$. If G has a dihedral subquotient D_{2qr} (order $2qr$) with distinct primes q and r and further that G contains distinct real G-conjugacy classes*

$(x)^{\pm}$, $(y)^{\pm}$ of elements x, y not of prime power order, then $\mathfrak{D}\mathfrak{S}(G)$ contains a direct summand of $\mathrm{RO}(G)$ *of rank 1.*

Theorems 2 and 3 follow from Theorem 6. In addition, we conclude the next.

Theorem 7. *Laitinen's Conjecture is affirmative for any finite nonsolvable gap group.*

References

[1] M. F. Atiyah and R. Bott, *A Lefshetz fixed point formula for elliptic complexes: II. Applications*, Ann. of Math. **88** (1968), 451–491.

[2] S. E. Cappell and J. L. Shaneson, *Fixed points of periodic maps*, Proc. Nat. Acad. Sci. USA **77** (1980), 5052–5054.

[3] S. E. Cappell and J. L. Shaneson, *Fixed points of periodic differentiable maps*, Invent. Math. **68** (1982), 1–19.

[4] S. E. Cappell and J. L. Shaneson, *Representations at fixed points*, Group Actions on Manifolds (Boulder, Colo., 1983), pp. 151–158, Contemp. Math., 36, Amer. Math. Soc., Providence, RI, 1985.

[5] GAP, *Groups, Algorithms, Programming, a System for Computational Discrete Algebra*, Release 4.3, 06 May 2002, URL: http://www.gap-system.org.

[6] X.M. Ju, *The Smith isomorphism question: A review and new results*, in RIMS Kokyuroku (2007): The Theory of Transformation Groups and Its Applications, eds. S. Kuroki and I. Nagasaki, Res. Inst. Math. Sci., Kyoto University, 2007.

[7] E. Laitinen and M. Morimoto, *Finite groups with smooth one fixed point actions on spheres*, Forum Math. **10** (1998), 479–520.

[8] E. Laitinen and K. Pawałowski, *Smith equivalence of representations for finite perfect groups*, Proc. Amer. Math. Soc. **127** (1999), 297–307.

[9] J. W. Milnor, *Whitehead torsion*, Bull. Amer. Math. Soc. **72** (1966), 358–426.

[10] M. Morimoto, *Equivariant surgery theory: Deleting-inserting theorems of fixed point manifolds on spheres and disks*, K-Theory **15** (1998), 13–32.

[11] M. Morimoto, *Equivariant surgery theory for homology equivalences under the gap condition*, Publ. Res. Inst. Math. Sci. Kyoto Univ. **42** (2006), 481–506.

[12] M. Morimoto, *The Smith problem and a counterexample to Laitinen's conjecture*, RIMS Kokyuroku no. 1517 (2006), Res. Inst. Math. Sci., Kyoto Univ., 25–31.

[13] M. Morimoto, *Smith equivalent* $\mathrm{Aut}(A_6)$*-representations are isomorphic*, accepted by Proc. Amer. Math. Soc., 2006.

[14] M. Morimoto, *Fixed point sets of smooth actions on spheres*, accepted by J. *K*-Theory, 2007.

[15] M. Morimoto and K. Pawałowski, *Smooth actions of Oliver groups on spheres*, Topology **42** (2003), 395-421.

[16] R. Oliver, *Fixed point sets of groups on finite acyclic complexes*, Comment. Math. Helv. **50** (1975), 155–177.

[17] B. Oliver, *Fixed point sets and tangent bundles of actions on disks and euclidean spaces*, Topology **35** (1996), 583–615.

[18] K. Pawałowski and R. R. Solomon, *Finite Oliver groups with rank integer* 0 *or* 1 *and Smith equivalence of group modules*, Algebr. Geom. Topol. **2** (2002), 843–895 (electronic).

[19] T. Petrie, *G surgery. I. A survey*, in Algebraic and geometric topology (Santa Barbara, Calif., 1977), pp. 197–233, Lecture Notes in Math., 664, Springer Verlag, Berlin-Heidelberg-New York, 1978.

[20] T. Petrie, *Pseudoequivalences of G-manifolds*, Algebraic and Geometric Topology (Proc. Symps. Pure Math., Stanford Univ., Stanford, Calif, 1976) Part 1, pp. 169–210, Proc. Symps. Pure Math., 32, Amer. Math. Soc., Providence, RI, 1978.

[21] T. Petrie, *Three theorems in transformation groups*, Algebraic Topology (Aarhus 1978), pp. 549–572, Lecture Notes in Math., 763, Springer Verlag, Berlin-Heidelberg-New York, 1979.

[22] T. Petrie, *The equivariant J homomorphism and Smith equivalence of representations*, Current Trends in Algebraic Topology (London, Ont., 1981), pp. 223–233, CMS Conf. Proc. 2, Part 2, Amer. Math. Soc., Providence, RI, 1982,

[23] T. Petrie, *Smith equivalence of representations*, Math. Proc. Cambridge Philos. Soc. **94** (1983), 61–99.

[24] T. Petrie and J. Randall, *Transformation Groups on Manifolds*, Marcel Dekker, Inc., New York and Basel, 1984.

[25] T. Petrie and J. Randall, *Spherical isotropy representations*, Publ. Math. IHES **62** (1985), 5–40.

[26] C. U. Sanchez, *Actions of groups of odd order on compact orientable manifolds*, Proc. Amer. Math. Soc. **54** (1976), 445–448.

[27] P. A. Smith, *New results and old problems in finite transformation groups*, Bull. Amer. Math. Soc. **66** (1960), 401–415.